AGT2024
Advances in General Topology

June 4-5, 2024

University of Messina

Link: Virtual room AGT2024

Room B-1-2, Edificio Didattico, University of Messina

Scientific Committee

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Program for June 4th (Afternoon session in Italy)

15:45-16:00 Wellcome to the Conference

Chair: Maddalena Bonanzinga

16:00-16:25 Jan van Mill, University of Amsterdam, The Netherlands
Nowhere constant families of maps and resolvability

16:30-16:55 Vladimir Tkachuk, Universidad Autónoma Metropolitana, Mexico
On nice embeddings of Lindelöf scattered spaces

17:00-17:25 Leandro F. Aurichi, University São Paulo, Brasil
Topological remarks on end and edge-end spaces

17:30-17:55 Nathan A. Carlson, California Lutheran University, USA
On diagonal degrees and star networks

18:00-18:25 Santi Spadaro, University of Palermo, Italy
Towers and cellular-Lindelof spaces

18:30-18:55 Rodrigo Hernández-Gutiérrez, Universidad Autónoma Metropolitana, Mexico
An application of scales to cellular-Lindelöf spaces

Program for June 5th (Morning session in Italy)

9:45-10:00 Wellcome to the Morning session

Chair: Davide Giacopello

10:00-10:25 M. Sakai, Kanagawa University, Japan,
Embeddable ultrafilters into the Pixley-Roy spaces over ultrafilters

10:30-10:55 Paul Gartside, Pittsburgh University, USA
"Chain Conditions" and Pixley-Roy/Ocan Spaces

11:00-11:25 Lyubomyr Zdomskyy, Technische Universität Wien, Austria
Combinatorial covering properties in the Sacks model

11:30-11:55 Piotr Szewczak, Cardinal Stefan Wyszyński University in Warsaw, Poland
Topological selections and products

12:00-12:25 Daniele Toller, Aalborg University, Denmark
A gentle introduction to cellular automata

12:30-12:55 Ivan Gotchev, Central Connecticut State University, USA
On a question of Bonanzinga

Abstracts

Jan van Mill: *Nowhere constant families of maps and resolvability*

Joint work with István Juhász

If X is a topological space and Y is any set then we call a family $\mathcal{F}$ of maps from X to Y *nowhere constant* if for every non-empty open set U in X there is $f \in \mathcal{F}$ with $|f[U]| > 1$, i.e. f is not constant on U. We prove the following result that improves several earlier results in the literature. If X is a topological space for which $C(X)$, the family of all continuous maps of X to $\mathbb{R}$, is nowhere constant and X has a π-base consisting of connected sets then X is $\mathfrak{c}$-resolvable.

Vladimir V. Tkachuk: *On nice embeddings of Lindelöf scattered spaces*

We will give an example of an Eberlein compact space K such that some Lindelöf subspace of K fails to be a Lindelöf Σ-space. We will also show that any scattered Lindelöf subspace of a σ-product of first countable spaces is σ-compact. The main result of this talk states that if X is the G_δ-modification of a scattered compact space, then $ext(C_p(X)) = \omega$.

Leandro F. Aurichi: *Topological remarks on end and edge-end spaces*

Joint work with with Paulo S. Magalhães Júnior and Lucas S. Real.

For an infinite graph G, there is a standard topological space called the end space of G. This space has several applications in graph theory and has a long history of work on it. A motivation for this space is to represent connections among vertices of the graph. Classically, these connections are thought in terms of sets of vertices, in the sense "how many vertices are needed to separate a set?". Diestel asked the question of what are the topological spaces that are the end space of some graph. This question was recently solved by Pitz. Here we study another natural space, presented in a similar fashion, where the motivation is to represent the connections in terms of sets of edges, instead of vertices. One of our results is the characterization of what are the topological spaces obtained this way. Another result is a topological game that helps in the characterization presented by Pitz.

Nathan A. Carlson: *On diagonal degrees and star networks*

Given an open cover $\mathcal{U}$ of a topological space X, we introduce the notion of a star network for $\mathcal{U}$. The associated cardinal function $sn(X)$, where $e(X) \leq sn(X) \leq L(X)$, is used to establish new cardinal inequalities involving diagonal degrees. We show $|X| \leq sn(X)^{\Delta(X)}$ for a T_1 space X, giving a partial answer to a long-standing question of Angelo Bella. Many further results are given using variations of $sn(X)$. One result has as corollaries Buzyakova's theorem that a ccc space with a regular G_δ-diagonal has cardinality at most $\mathfrak{c}$, as well as three results of Gotchev. Further results lead to logical improvements of theorems of Basile, Bella, and Ridderbos, a partial solution to a question of the same authors, and a theorem of Gotchev, Tkachenko, and Tkachuk. Finally, we define the Urysohn extent $Ue(X)$ with the property $Ue(X) \leq \min\{aL(X), e(X)\}$ and use the Erdős-Rado theorem to show that $|X| \leq 2^{Ue(X)\overline{\Delta}(X)}$ for any Urysohn space X.

Santi Spadaro: *Towers and cellular-Lindelof spaces*

Joint work with Rodrigo Hernández-Gutiérrez.

Let κ be a cardinal. A space X is said to be (almost) *cellular-Lindelöf* if, for every κ-sized family $\mathcal{U}$ of pairwise disjoint non-empty open subsets of X, there is a Lindelöf subspace L of X such that L has non-empty intersection with every member of $\mathcal{U}$ (respectively, with κ-many members of $\mathcal{U}$). Cellular-Lindelöf spaces are an interesting common generalization of Lindelöf spaces and spaces with the countable chain condition, that was originally motivated by the problem of finding a common extension to Arhangel'skii's Theorem and the Hajnal-Juhász inequality (see [3]). While solving this problem required a shift in perspective (see [5]), the question of whether every cellular-Lindelöf first-countable regular space has cardinality at most continuum is still open, and various partial answers to it have recently appeared in the literature (see, for example, [2, 4, 7, 8]).

After giving an introduction to cellular-Lindelöf spaces, we will present two new examples regarding this class of spaces, the first one of which solves a question of Ofelia Alas, Luis Enrique Gutiérrez-Dominguez and Richard Wilson [1].

1. A consistent example of a normal almost cellular-Lindelöf space which is neither cellular-Lindelöf nor weakly Lindelöf.
2. A ZFC example of a space whose cellular Lindelöf property is independent of ZFC (and whose normality also turns out to be independent of ZFC).

The first example uses a tower of uncountable subsets of ω_1.

[1] O.T. Alas, L.E. Gutiérrez-Domínguez and R.G. Wilson, *Compact productivity of Lindelöf-type properties*, Acta Math. Hung. **167** (2022), 648–560.

[2] A. Bella, *On cellular-compact and related spaces*, Topology Appl. **281** (2020), Article ID 107203.

[3] A. Bella and S. Spadaro, *On the cardinality of almost discretely Lindelöf spaces*, Monatsh. Math. **186** (2018), 345–353.

[4] A. Bella and S. Spadaro, *Cardinal invariants of cellular-Lindelöf spaces*, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. RACSAM **113** (2019), 2805–2811.

[5] A. Bella and S. Spadaro, *A common extension of Arhangel'skii's Theorem and the Hajnal-Juhász's inequality*, Can. Math. Bull. **63** (2020), 197–203.

[6] A. Dow and R.M. Stephenson, *Productivity of cellular-Lindelöf spaces*, Topology Appl. **290** (2021), Article ID 107606.

[7] I. Juhász, L. Soukup and Z. Szentmiklóssy, *On cellular-compact spaces*, Acta Math. Hung. **162** (2020), 549–556.

[8] V.V. Tkachuk and R.G. Wilson, *Cellular-compact spaces and their applications*, Acta Math. Hung. **159** (2019), 674–688.

Rodrigo Hernández-Gutiérrez: *An application of scales to cellular-Lindelöf spaces*

Joint work with Santi Spadaro.

The class of cellular-Lindelöf spaces, which generalizes both the classes of Lindeöf spaces and spaces with the countable chain condition, has been recently studied by several authors. In this talk, I will present an example of a cellular-Lindelöf space answering an open question by Alas, Gutiérrez and Wilson from 2022. This example can be constructed in models of set theory where there exists a type of set called scale so I will also talk about the existence of scales and other applications they have.

Masami Sakai: *Embeddable ultrafilters into the Pixley-Roy spaces over ultrafilters*

For a T_1-space X, we denote by $PR(X)$ the Pixley-Roy space over X. For $p \in \omega^*$, let $X_p = \{p\} \cup \omega$ be the subspace of the Stone-Čech compactification $\beta\omega$ of the discrete space ω. we can show: if X_q can embedded into $PR(X_p)$ and X_p can be embedded into $PR(X_q)$, then X_p and X_q are homeomorphic (i.e., p and q are type-equivalent). So, it is natural to ask when X_q can be embedded into $PR(X_p)$. For this question, we can see: If p is selective, then $PR(X_p)$ contains copies of some $X_{q_n} (n \in \mathbb{N})$ which are pairwise non-homeomorphic.

Paul Gartside: *"Chain Conditions" and Pixley-Roy/Ocan Spaces*

The "countable chain condition" (every uncountable family of open sets contains two that meet) and "calibre ω_1" (every uncountable family of open sets contains an uncountable subfamily with non-empty intersection) are typical examples of "chain conditions". We discuss what it means to say that two spaces have the same chain conditions, and see that any two separable spaces have the same chain conditions. With the aid of Pixley-Roy/Ocan spaces we show there is a $2^{\mathfrak{c}}$ sized family of spaces of density no more than $\mathfrak{c}$ all of which have different chain conditions.

Lyubomyr Zdomskyy: *Combinatorial covering properties in the Sacks model*

By a *space* we mean a metrizable separable space. A space X is *Menger* if for any sequence $\mathcal{U}_0, \mathcal{U}_1, \dots$ of open covers of X, there are finite families $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \dots$ such that the family $\bigcup_{n\in\omega} \mathcal{F}_n$ covers X. If, moreover, the $\mathcal{F}_n$'s can be chosen in such a way that for every $x \in X$, $x \in \bigcup \mathcal{F}_n$ holds for all but finitely many n, X is said to be *Hurewicz*. We call a space *totally imperfect* if it contains no copy of the Cantor space. We shall discuss how combinatorics of Sacks conditions developed by Miller [2] combined with game characterization of the Menger property, allow to show that there are no totally imperfect Menger sets of reals of size continuum in the Sacks model. Therefore, the Menger property behaves in the Sacks model as an instance of the Perfect Set Property, sets are either small or contain a perfect set. For models, which satisfy that the dominating number has size continuum, there is always a totally imperfect Menger set of size continuum. (There are also models with small dominating number, where such sets exist.) Thus, combined with our result the existence of totally imperfect Menger sets of reals of size continuum is independent from ZFC.

Consonant spaces were introduced by Dolecki, Greco and Lechicki in 1995 and for the case $X \subseteq 2^\omega$ charaterized by Jordan [1] using a topological game on the complement $2^\omega \setminus X$. By considering a grouped version of the Menger game and using a similar approach like for the Menger space result, we conclude that every consonant and every Hurewicz subspace of 2^ω, as well as their complements, can be written as the union of ω_1-many compact sets in the Sacks model. In particular, there are only continuum many consonant spaces and Hurewicz spaces in this model.

[1] F. Jordan, *Consonant spaces and topological games,* Topology and its Applications **274** (2020), 107121.

[2] A. Miller, *Mapping a set of reals onto the reals,* Journal of Symbolic logic **48** (1983), 575–584.

Piotr Szewczak: *Topological selections and products*

Joint work with Boaz Tsaban and Lyubomyr Zdomskyy.

A topological space X is *Menger* if for every sequence of open covers $\mathcal{O}_1, \mathcal{O}_2, \ldots$ there are finite sets $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that the family $\{\bigcup \mathcal{F}_1, \bigcup \mathcal{F}_2, \ldots\}$ is a cover of X. If we can request that for every element $x \in X$ the set $\{n : x \in \bigcup \mathcal{F}_n\}$ is co-finite, then the space X is *Hurewicz.* The above properties generalize σ-compactness and the Hurewicz property is strictly stronger than Menger. We consider products of Menger or Hurewicz spaces in various models of set theory and their connections to products of spaces with covering properties based on similar patterns as mentioned above.

The research was funded by the National Science Center, Poland and the Austrian Science Found under the Weave-UNISONO call in the Weave programme project: *Set-theoretic aspects of topological selections* 2021/03/Y/ST1/00122.

Daniele Toller: *A gentle introduction to cellular automata*

In this talk I will give a brief introduction to Cellular Automata, from their introduction in the 1940s by John von Neumann, to their popularization in the 1970s with John Conway's Game of Life, and then some attempt at their classification by Stephen Wolfram in the 1980s. If time permits, I will mention some recent results obtained in collaboration with H. Akin, D. Dikranjan, A. Giordano Bruno on linear and one-dimensional cellular automata.

Ivan Gotchev: *On a question of Bonanzinga*

In 1969, Arhangel′skiĭ proved in [1] that if X is a Hausdorff space, then $|X| \leq 2^{\chi(X)L(X)}$, where $\chi(X)$ is the character and $L(X)$ is the Lindelöf degree of X . Since then it has been an open question if his inequality is true for every T_1-space X. In 2014, we proved in [3] that if X is a T_1-space, then $|X| \leq nh(X)^{\chi(X)L(X)}$, where $nh(X)$ is the non-Hausdorff number of X. In that way we were able to positively answer this question for every T_1-space for which $nh(X) \leq 2^{\chi(X)L(X)}$, and, in particular, when $nh(X)$ is not grater than the cardinality of the continuum. A simple example shows that our inequality is not always true for T_0-spaces.

Arhangel′skiĭ and Šapirovskiĭ strengthened Arhangel′skiĭ's inequality in 1974 by showing that if X is a Hausdorff space, then $|X| \leq 2^{t(X)\psi(X)L(X)}$, where $t(X)$ is the tightness and $\psi(X)$ is the pseudocharacter of X.

In 2013, Bonanzinga asked if Arhangel′skiĭ– Šapirovskiĭ's inequality is true for every T_1-space with finite Hausdorff number $H(X)$. We recall that if $n \geq 2$ is an integer and X is a topological space, then $H(X) = n$ iff $nh(X) = n$.

In this talk we will show how Arhangel′skiĭ– Šapirovskiĭ's inequality, and therefore, Arhangel′skiĭ's inequality, could be extended to be valid for all topological spaces. It follows from this result that if X is a T_1-space such that $nh_s(X)$ is finite, then $|X| \leq 2^{t(X)\psi(X)L(X)}$. Unfortunately, this result only partially answers Bonanzinga's question because $nh(X) \leq nh_s(X)$ for every topological space X. (We recall that the *non-Hausdorff number of a space X with respect to singletons*, denoted by $nh_s(X)$, is $nh_s(X) := 1 + \sup\{|cl_\theta(\{x\}) : x \in X\}$ [4]).

[1] *The power of bicompacta with first axiom of countability,* (Russian), Dokl. Akad. Nauk SSSR **187** (1969), 967–970.

[2] Maddalena Bonanzinga, *On the Hausdorff number of a topological space*, Houston J. Math. **39** (2013), no. 3, 1013–1030.

[3] Ivan S. Gotchev, *The non-Hausdorff number of a topological space*, Topology Proc. **44** (2014), 249–256.

[4] Ivan S. Gotchev, *Generalizations of two cardinal inequalities of Hajnal and Juhász*, Topology Appl. **221** (2017), 425–431.